Introduction to Wavelet Analysis

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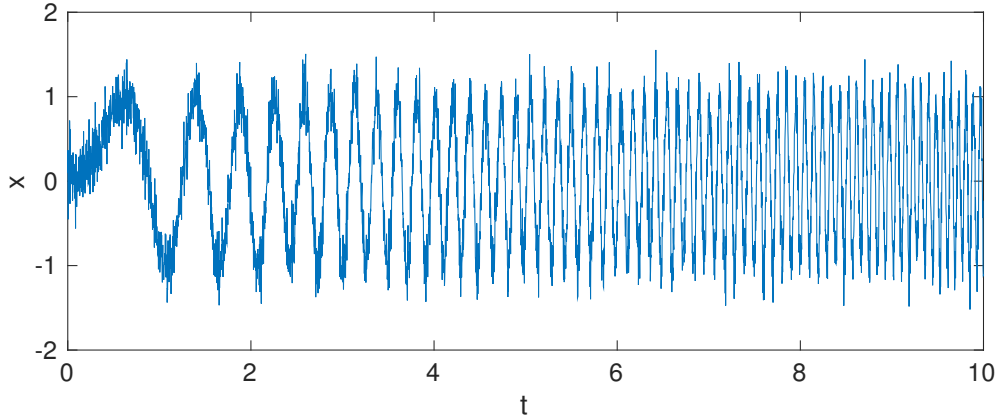
Content

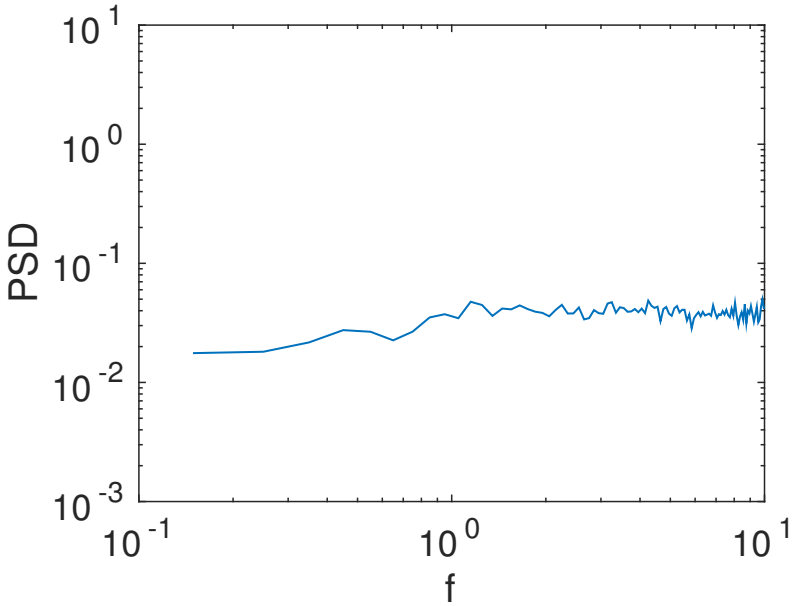
- Why use another time-frequency analysis?
- Basic concept of Wavelets
- Wavelet spectrum
- Cross Wavelet transform and coherence
- Significance

Why use another time-frequency analysis?

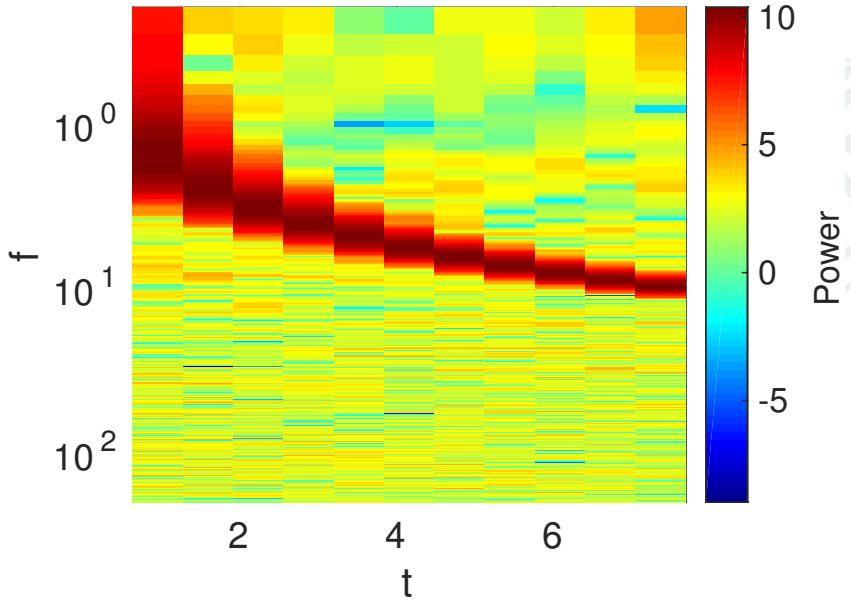
- advantages of Fourier analysis
  - easy to apply
  - very fast code implementation
- down sides of Fourier analysis
  - time information gets lost
  - assuming the signal to be periodic
- How to get information on specific frequency/event at specific time?



Global PSD from FT

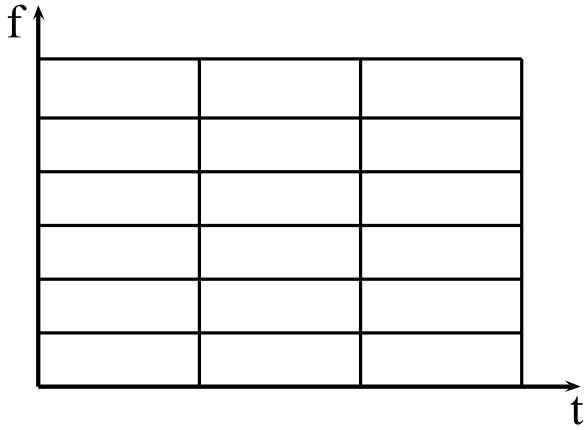


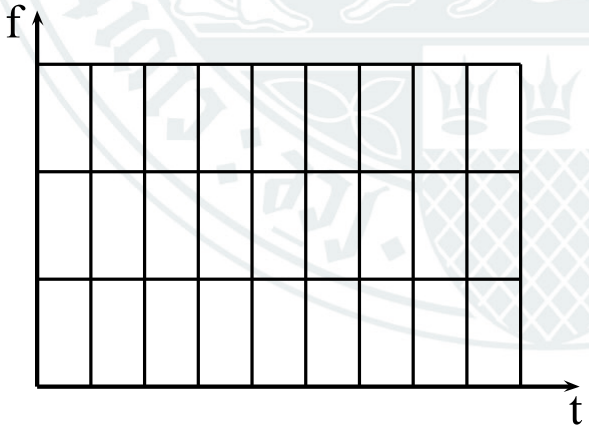
Spectrogram STFT



Characteristics of Fourier Analysis

- dependent on the choice of window (taper) that eliminates leakage effects
- constant time and frequency resolution
- uncertainty of ideal window size
  - small window $\rightarrow$ good time resolution, bad frequency resolution
  - large window $\rightarrow$ good frequency resolution, bad time resolution





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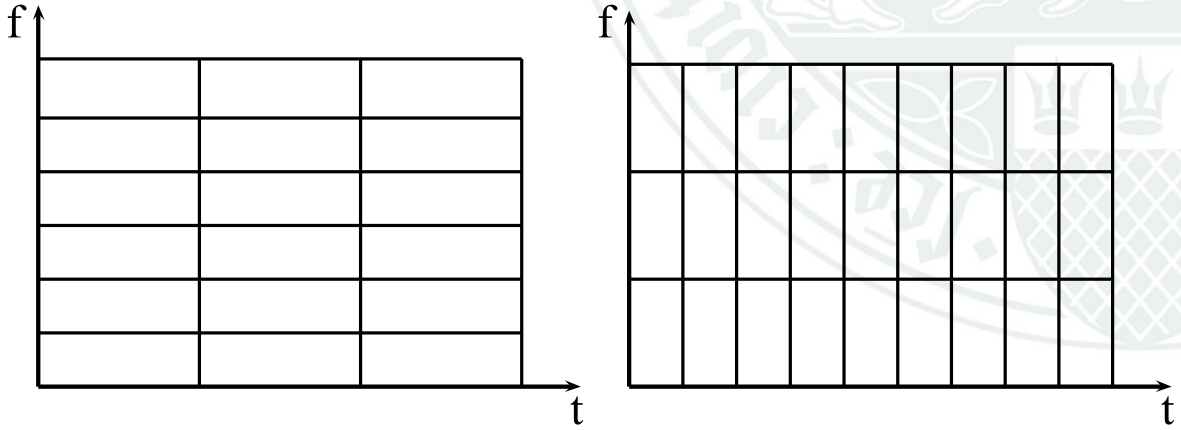
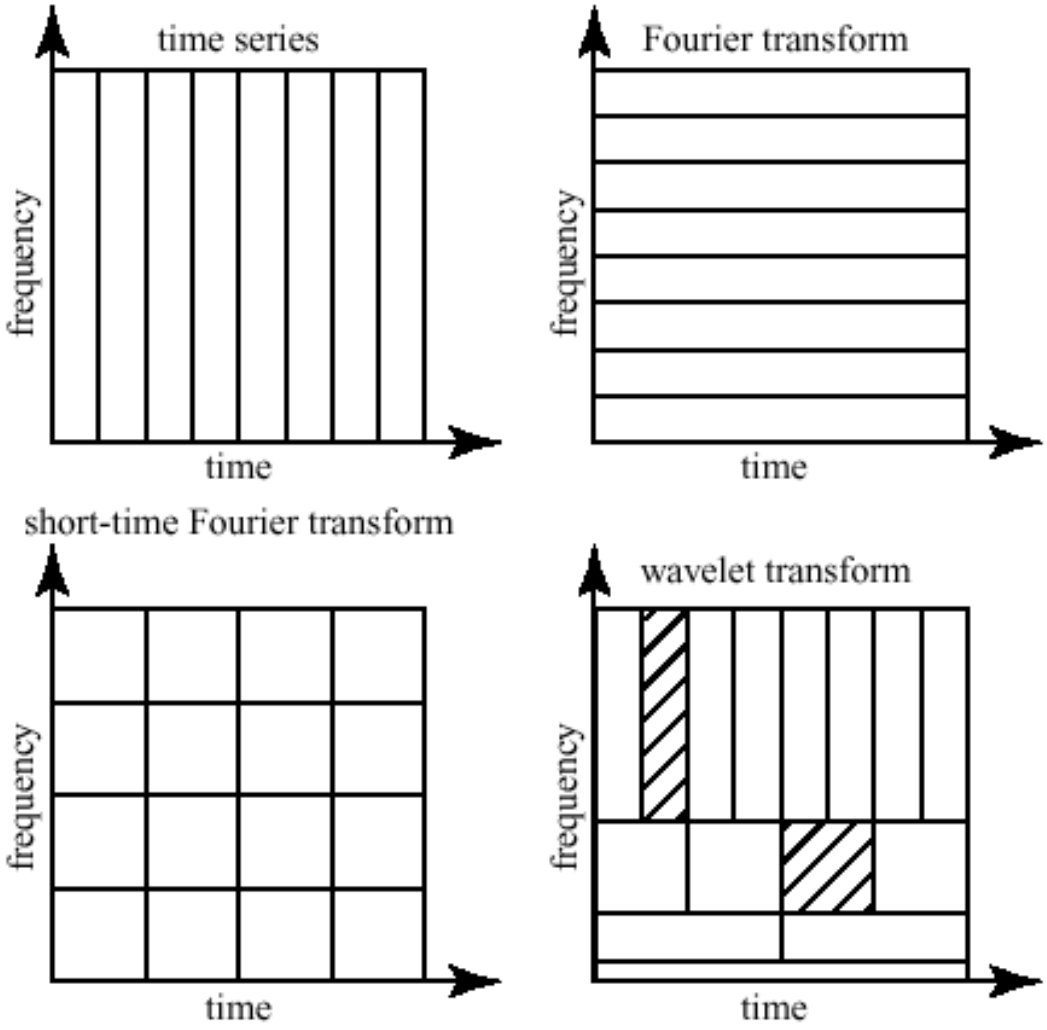


Wavelet analysis with frequency-dependent window sizes

Time-frequency resolutions

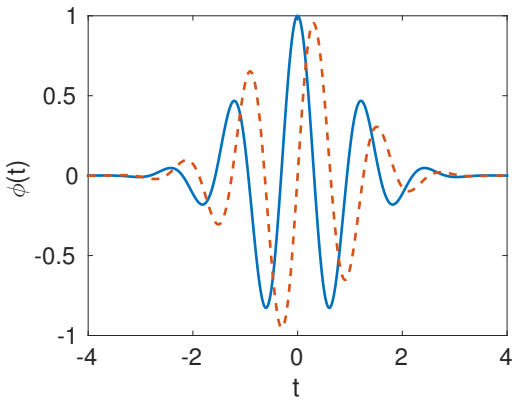


Wavelet transform

The **Wavelet transform** is the convolution of a signal $x(t)$ with the so called "Wavelet mother function" $\psi(t)$

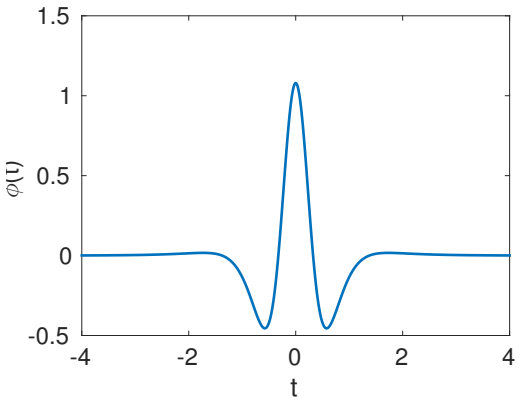
$$W_x(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{\infty} x(t)\, \psi^* \left(\frac{t-\tau}{s} \right) dt$$

Morlet Wavelet

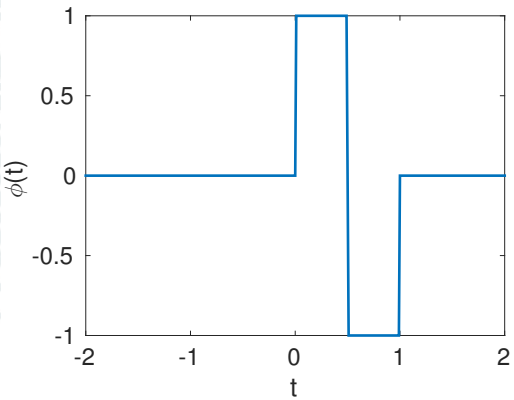


$$\psi(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-\frac{t^2}{2}}$$

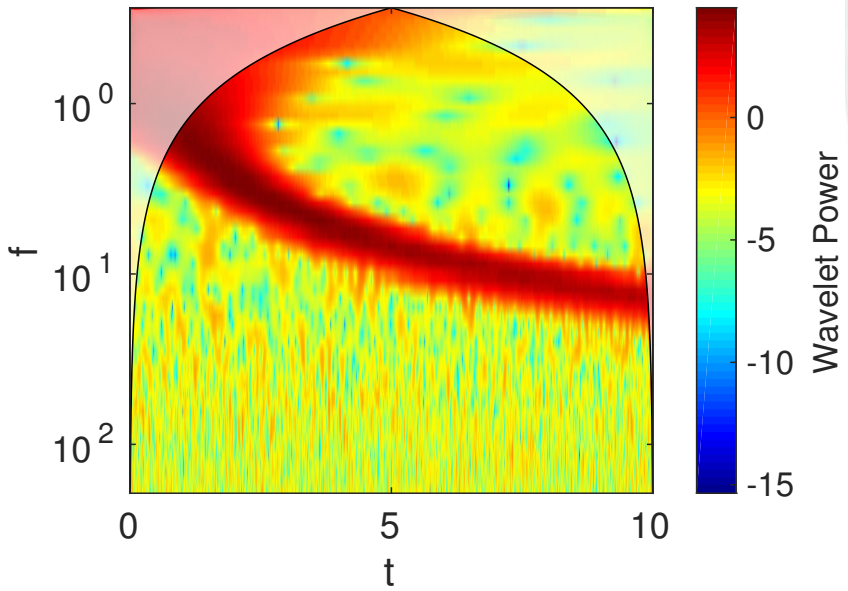
Paul Wavelet

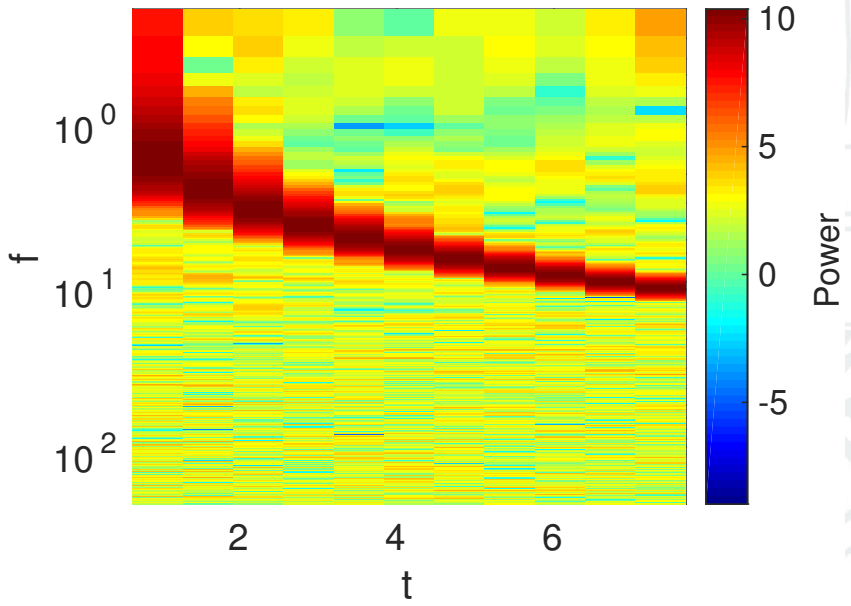


Haar Wavelet

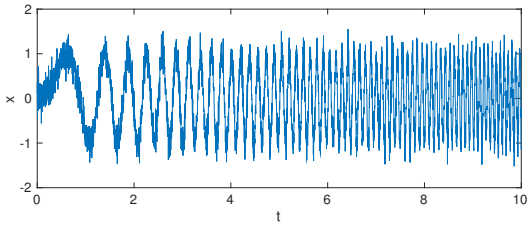


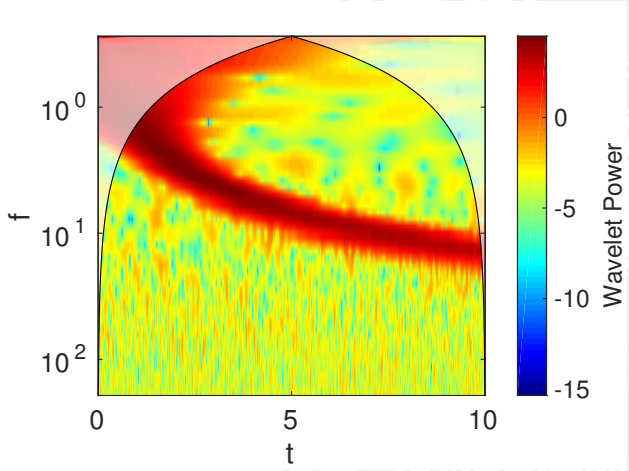
Wavelet spectrum

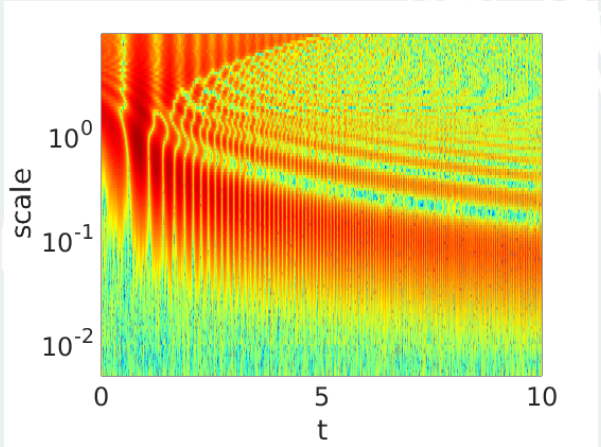


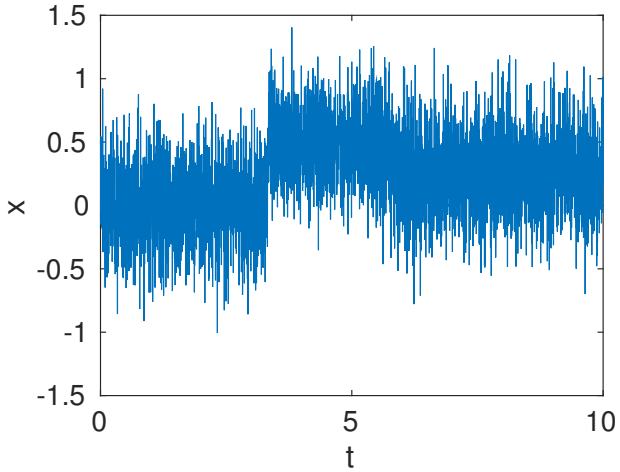


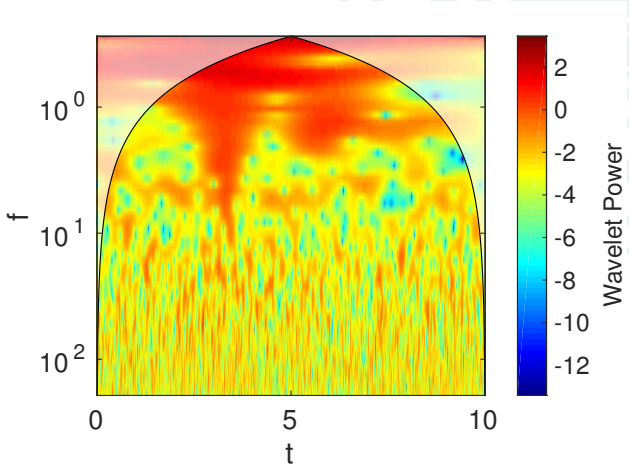
Choice of Wavelet

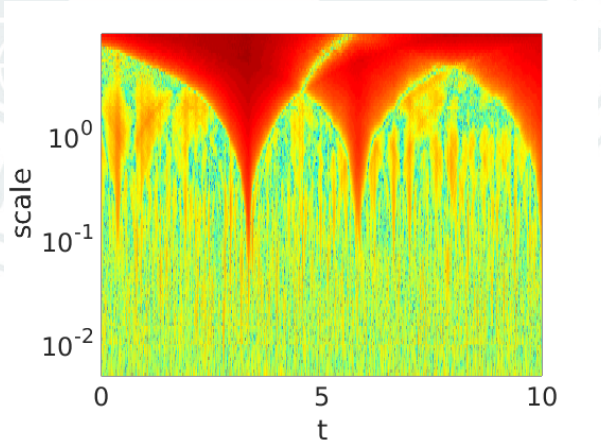




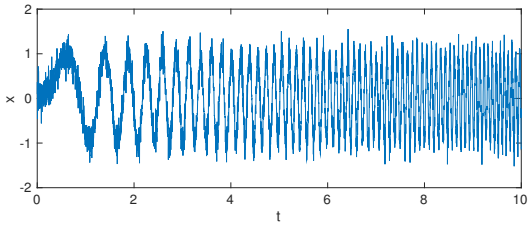


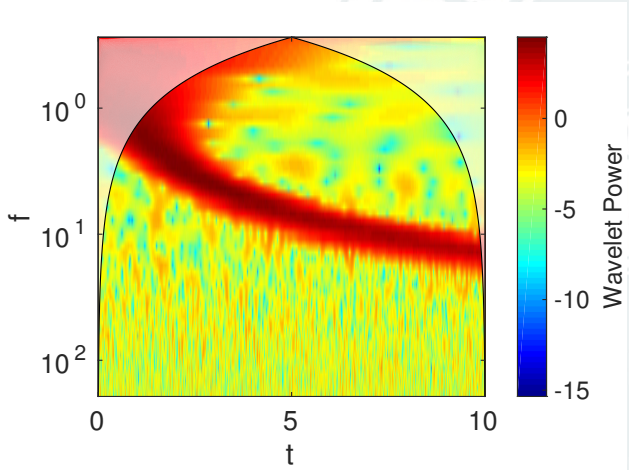


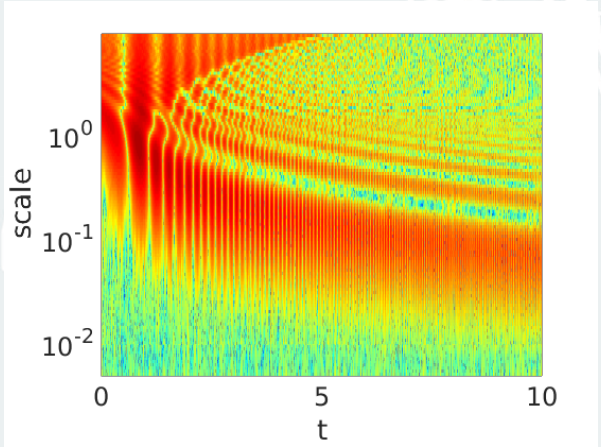


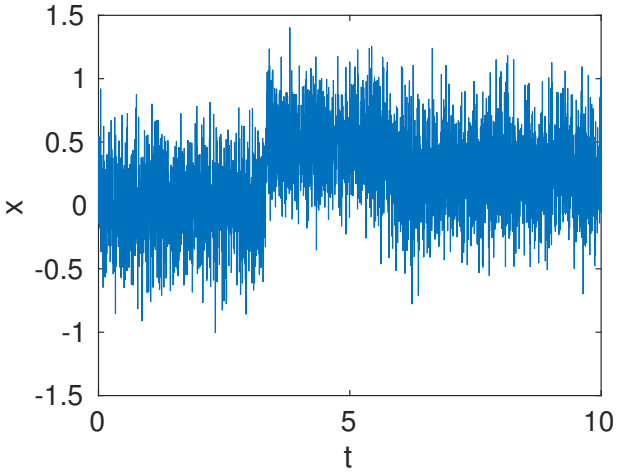


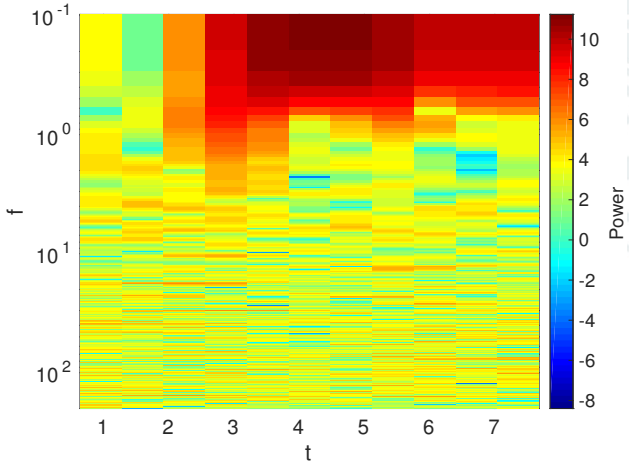
Choice of Wavelet

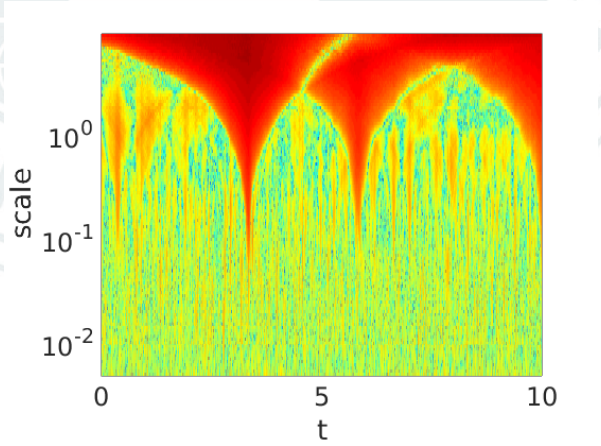






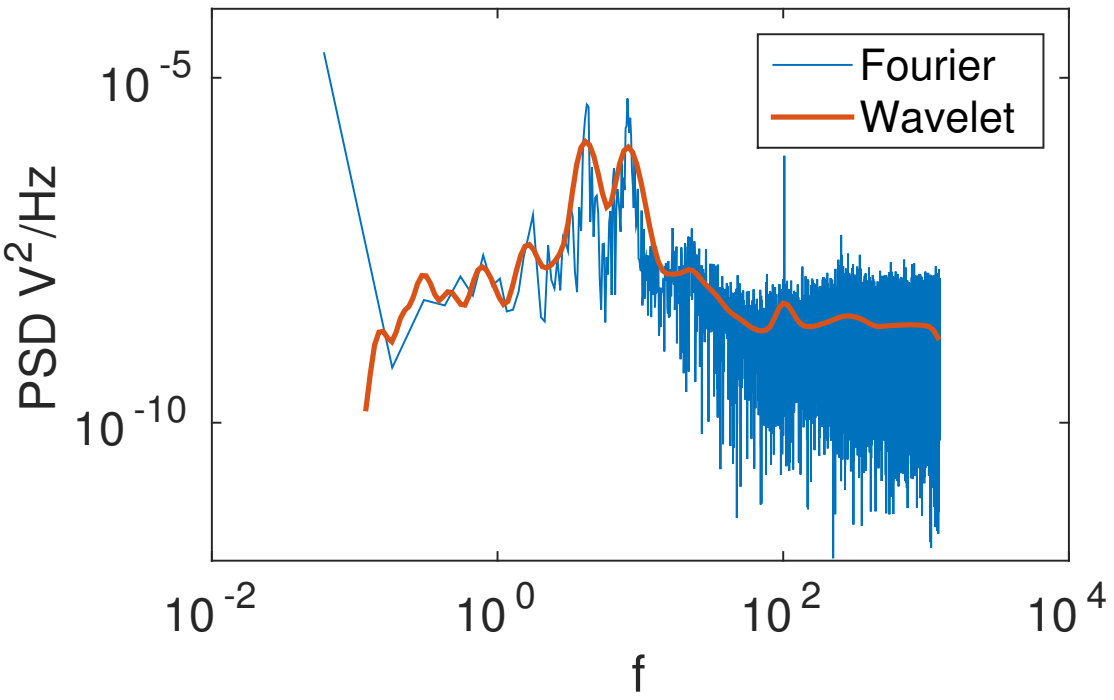


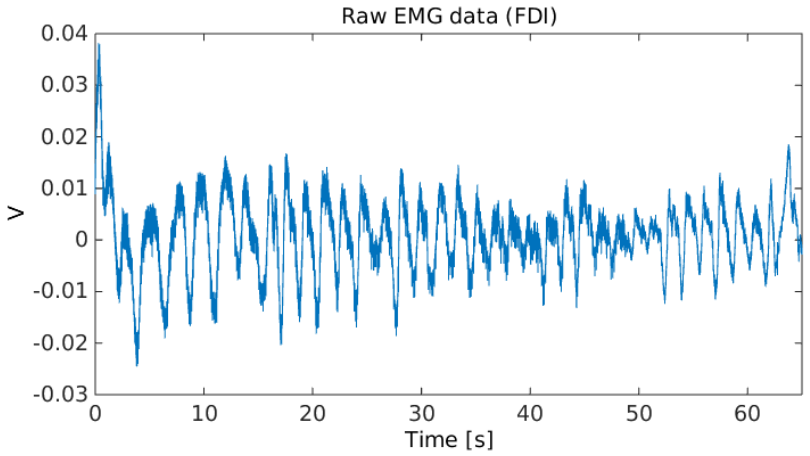


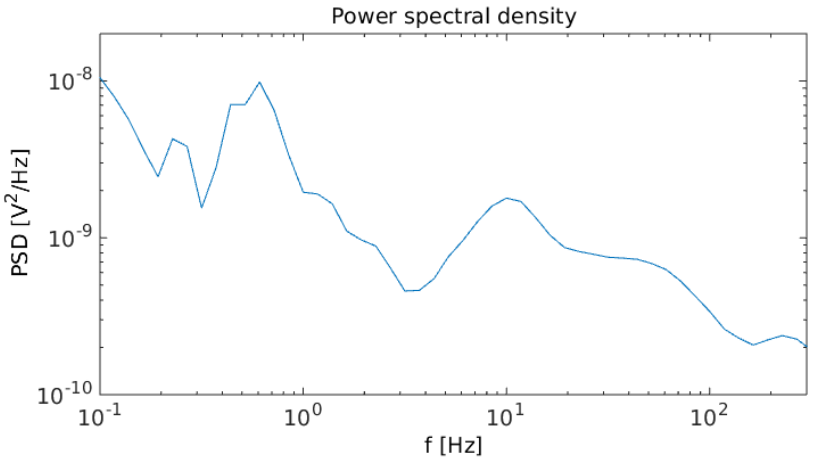


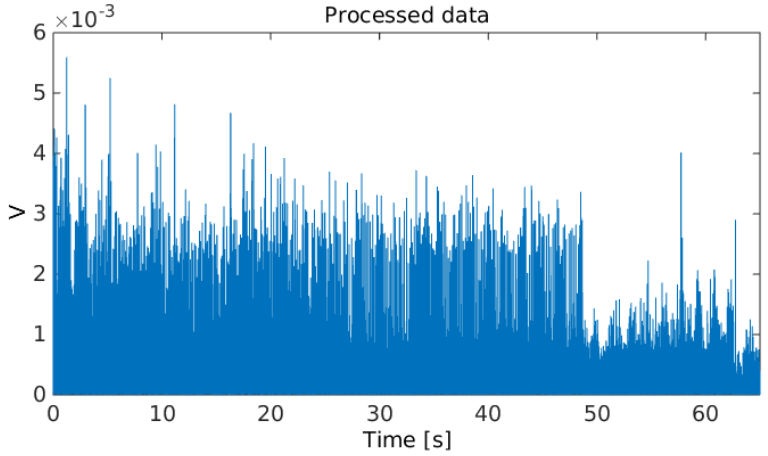
Global wavelet spectrum

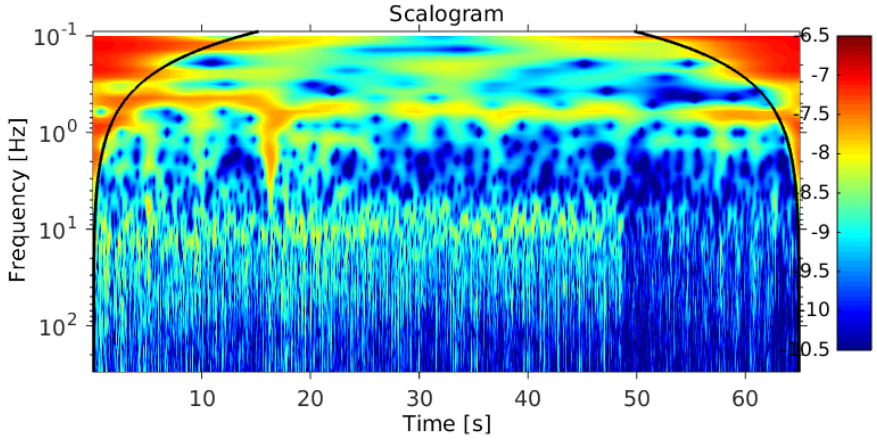
$$P(f) = \frac{2\delta t}{N} \sum_{i=0}^{N} W(t_i, f)$$

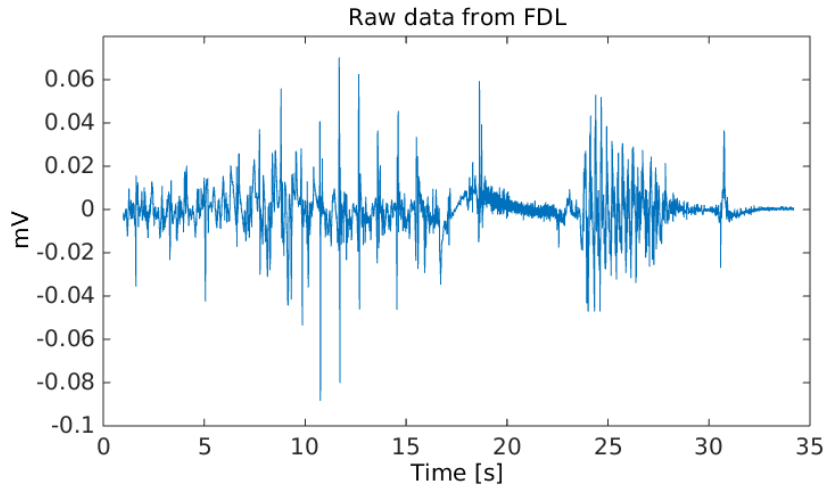


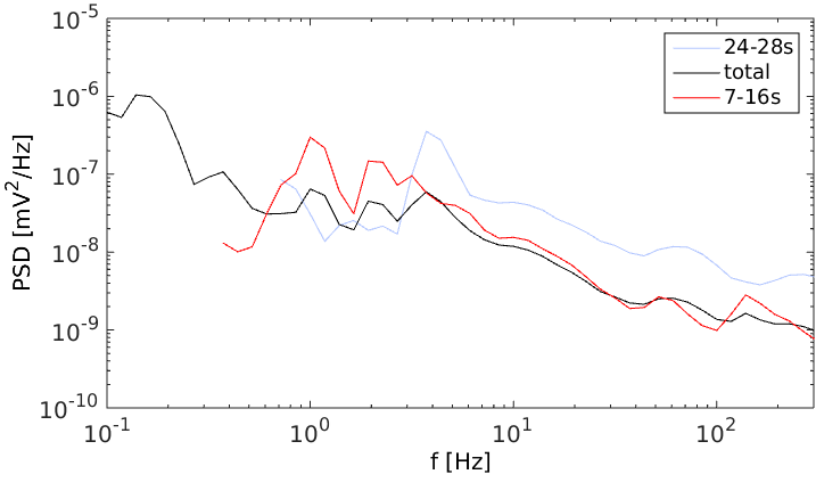

Raw EMG data (FDI)
V
Time [s]

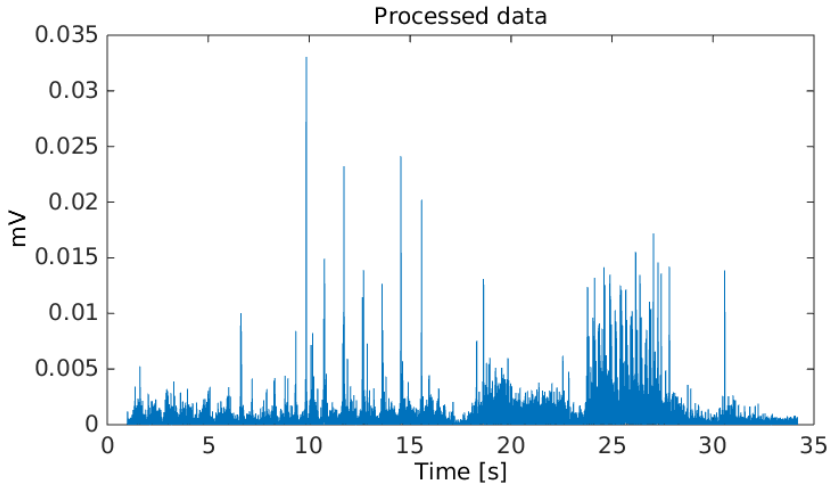

Power spectral density
PSD [V²/Hz]
f [Hz]

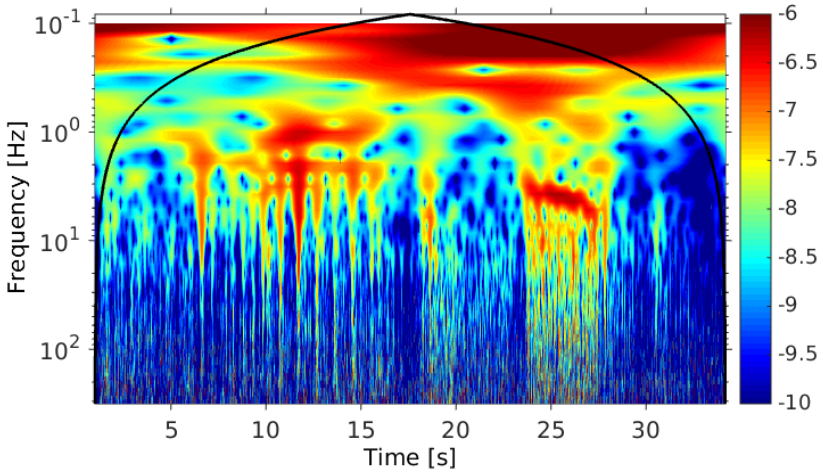

Processed data
×10⁻³
V
Time [s]


Scalogram
Frequency [Hz]
Time [s]


Raw data from FDL
mV
Time [s]


24-28s
total
7-16s
PSD [mV^2/Hz]
f [Hz]


Processed data
mV
Time [s]


Frequency [Hz]
Time [s]

Cross Wavelet transform

The cross Wavelet transform is defined as

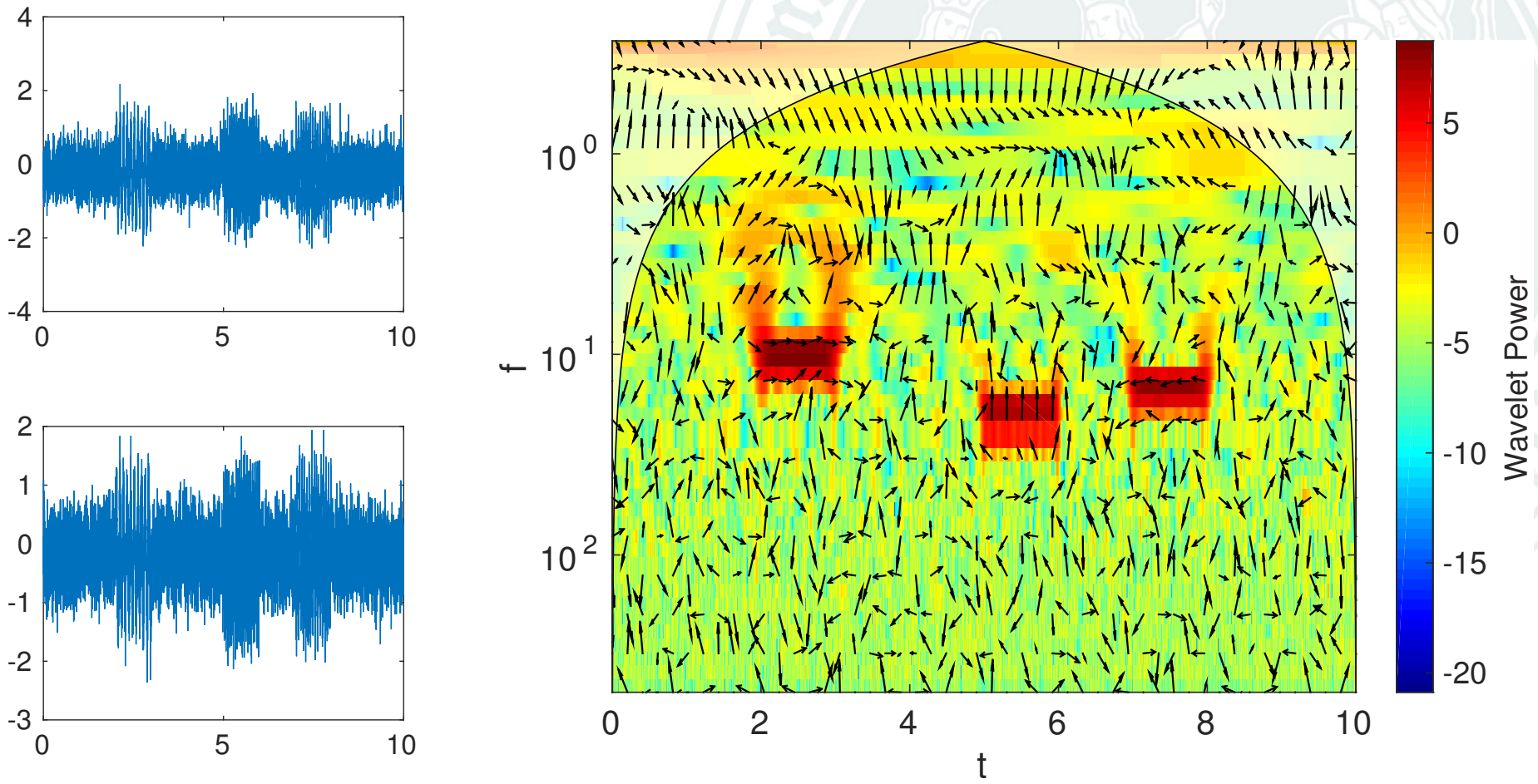
$$W_{xy}(t,s) = W_x(t,s)W_y^*(t,s)$$

Wavelet phase

$$\phi(t,s) = \arctan\left(\frac{\mathrm{Im}(W_{xy})}{\mathrm{Re}(W_{xy})}\right)$$

$\rightarrow$ time/frequency phase allows further analysis

Cross Wavelet transform

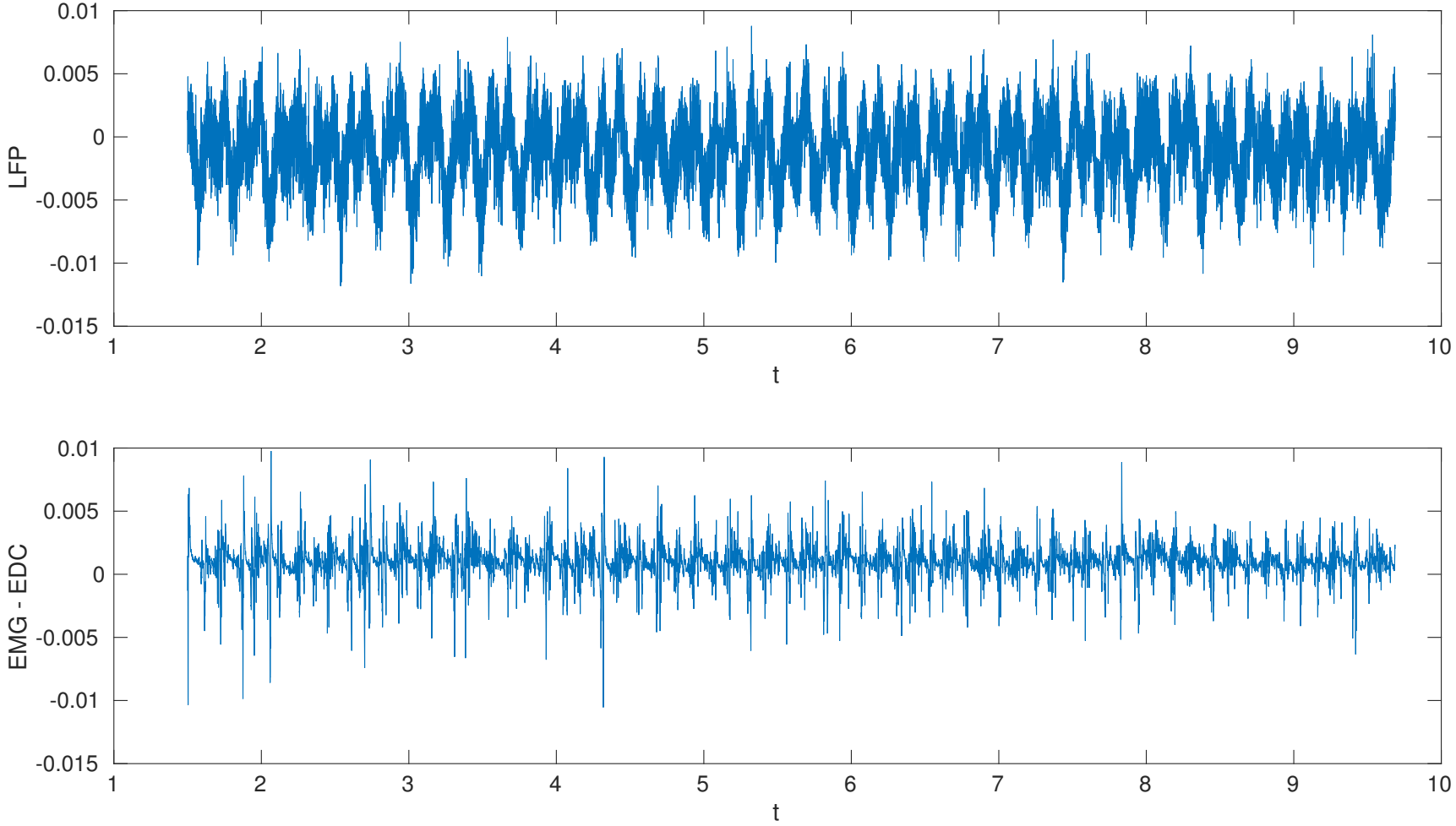


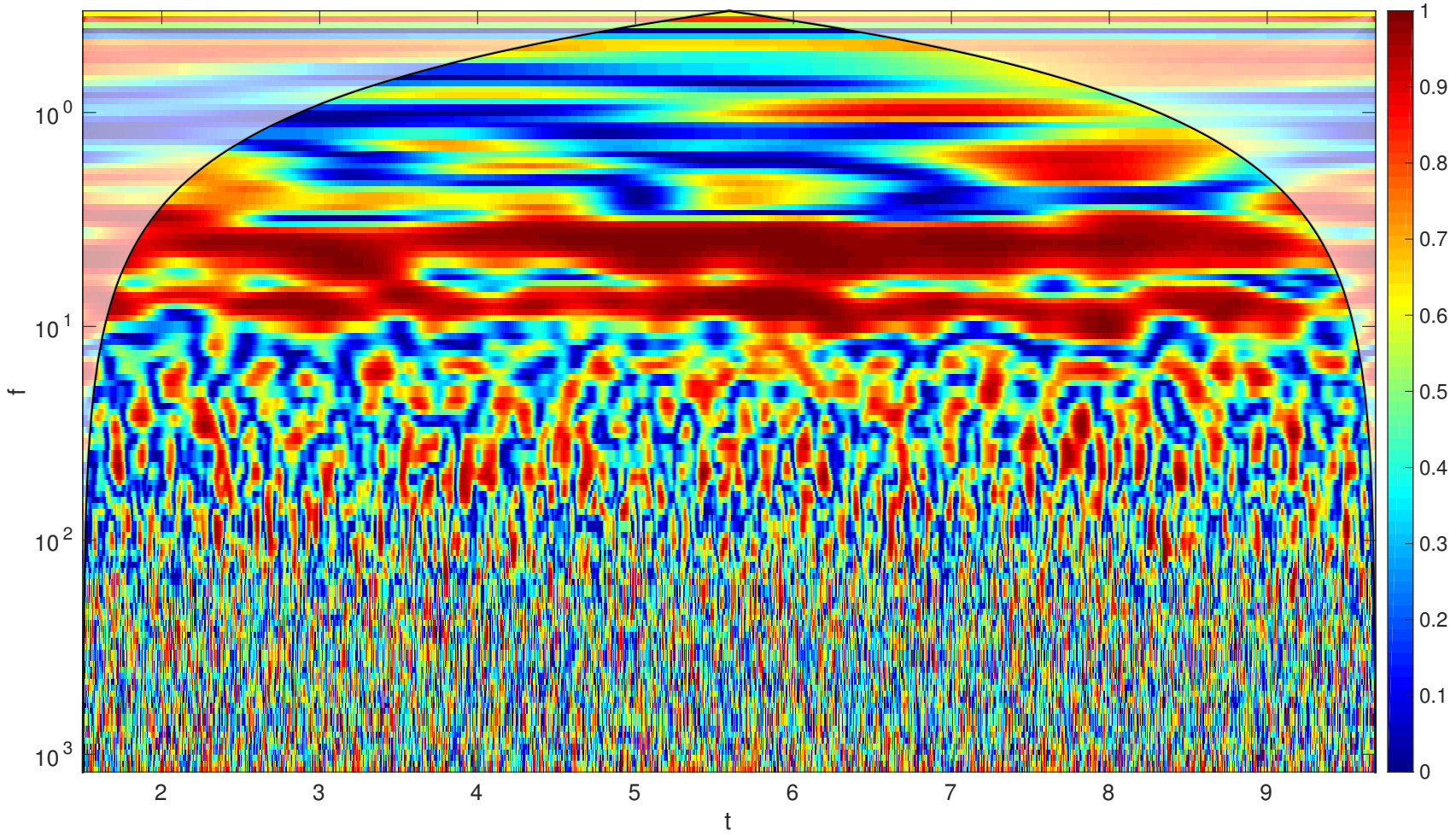
Wavelet coherence

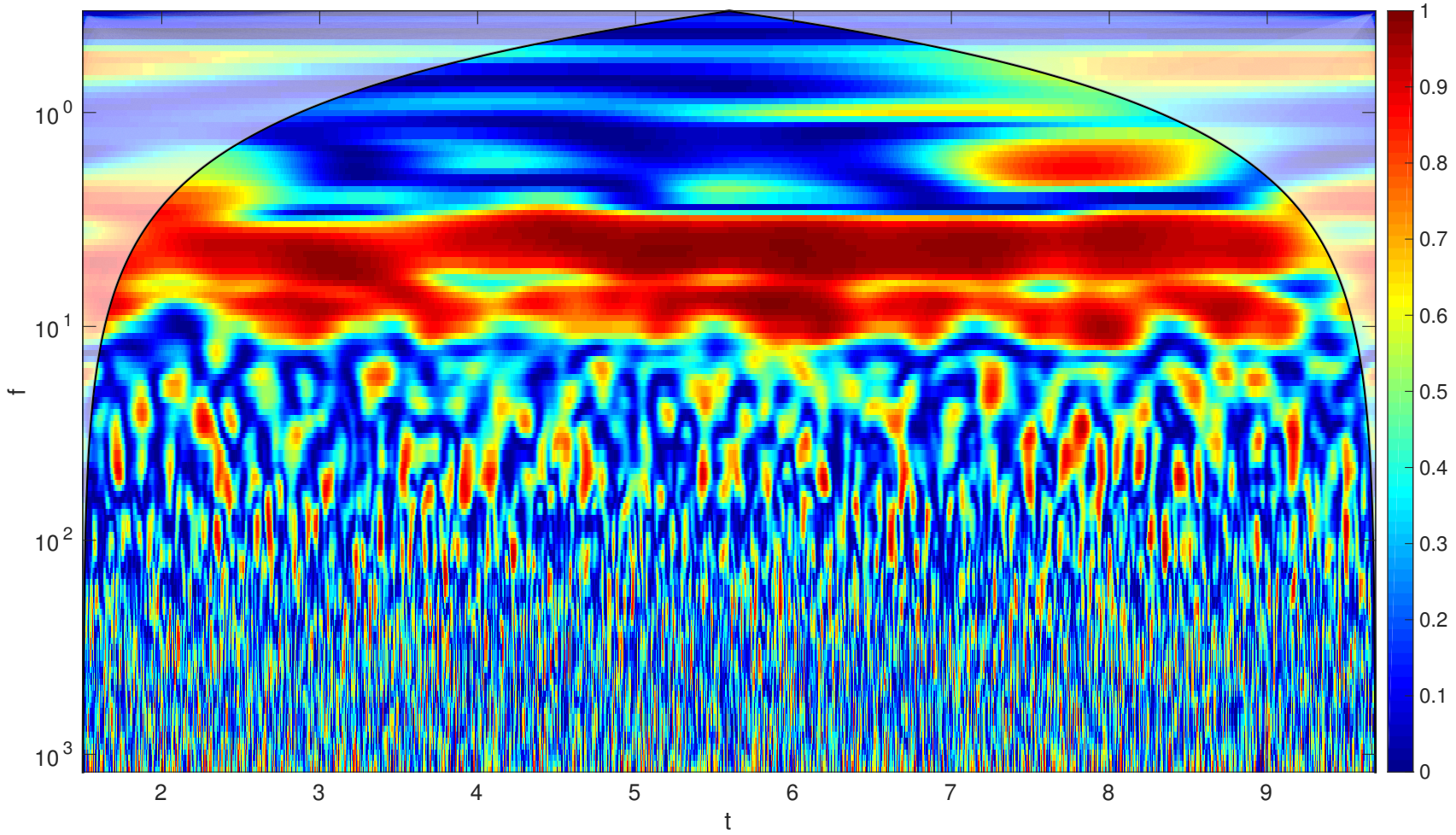
$$R^2(t,s) = \frac{|\langle W_{xy}(t,s)\rangle|^2}{\left\langle |W_{xx}(t,s)|^2 \right\rangle \left\langle |W_{yy}(t,s)|^2 \right\rangle}$$

where $\langle \cdot \rangle$ denotes the smoothing operator

- smoothing is frequency dependent
- smoothing in time (Gaussian Window for Morlet)
- additional smoothing over scales/frequencies optional


0.01
0.005
0
-0.005
-0.01
-0.015
LFP
1
2
3
4
5
6
7
8
9
10
t
0.01
0.005
0
-0.005
-0.01
-0.015
EMG - EDC
1
2
3
4
5
6
7
8
9
10
t


f
10^0
10^1
10^2
10^3
2
3
4
5
6
7
8
9
t
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0


f
10^0
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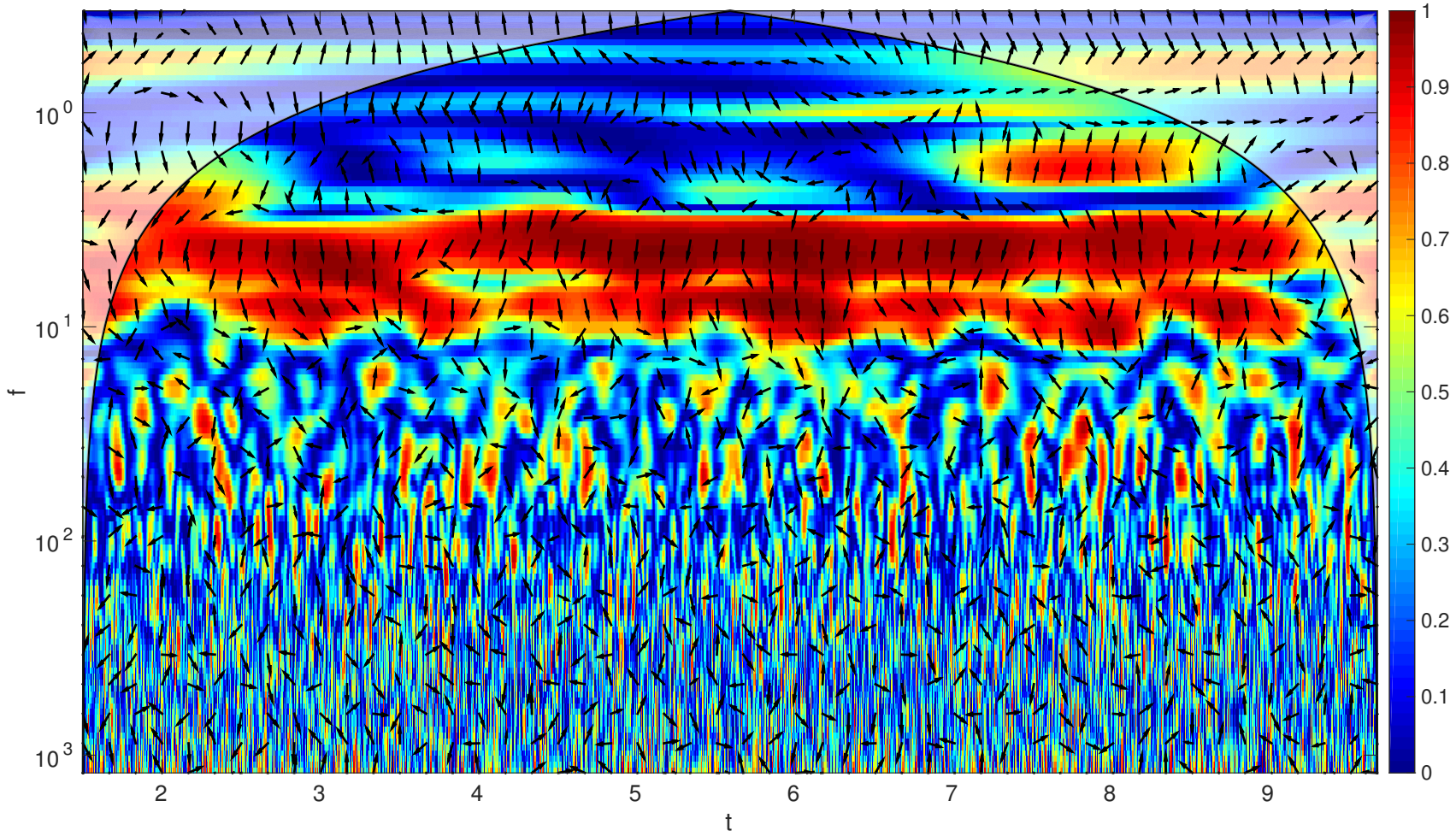
f
10^0
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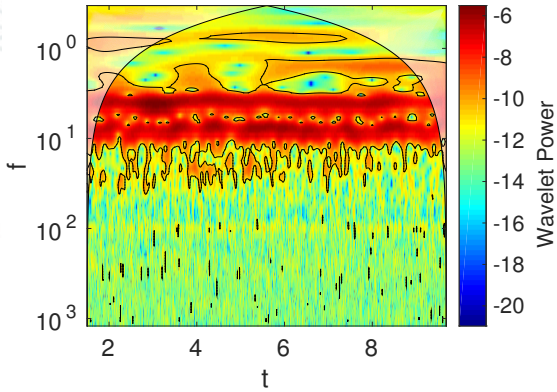
9
t
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Significance

- Significance of coherence $R^2(t,s)$
  1) $r_\alpha = 1 - \alpha^{1/(n-1)}$ where n is the number of measurements and e.g. $\alpha = 0.05$ (Bigot et al., 2011)
  2) making significance dependent on smoothing operator (Lachaux et al., 2002)
  3) using Monte Carlo simulation for given AR(x) process



- significance of cross spectrum:
  - Bigot et al. (2011) suggest self developed significance for cross Wavelet spectrum

Summary

- Wavelets provides scale/frequency optimized analysis with time resolution
- no "a priori" choice of window size
- spikes and artefacts are visible/detectable in scalograms
- frequency and time dependent coherence
- phase with time and frequency resolution
- remaining questions:
  - What are suitable Wavelet functions for EMG/EEG/LFP?
  - Which method should be used to evaluate significance?

Thank you for your attention!

Literature

Christopher Torrence and GP Compo. **A practical guide to wavelet analysis**. *Bulletin of the American Meteorological Society*, 1998.

Lachaux et al. **Estimating the time-course of coherence between single-trial brain signals: an introduction to wavelet coherence**. *Neurophysiologie Clinique/Clinical Neurophysiology*, 2002.

Bigot et al. **A new statistical test based on the wavelet cross-spectrum to detect time-frequency dependence between non-stationary signals: application to the analysis of cortico-muscular interactions.** *NeuroImage*, April 2011.